Response to Harmonic Excitation

Part 1 : Undamped Systems

Harmonic excitation refers to a sinusoidal external force of a certain frequency applied to a system. The response of a system to harmonic excitation is a very important topic because it is encountered very commonly and also covers the concept of resonance. Resonance occurs when the external excitation has the same frequency as the natural frequency of the system. It leads to large displacements and can cause a system to exceed its elastic range and fail structurally. A popular example, that many people are familiar with, is that of a singer breaking a glass by singing. Harmonic excitation is also commonly observed in systems that contain a rotating mass for example tires, engines, rotors, etc. This module develops the equations for the response of a single-degree-of-freedom system without damping to harmonic excitation using a spring-mass model. The next module will continue by including damping in the system model.
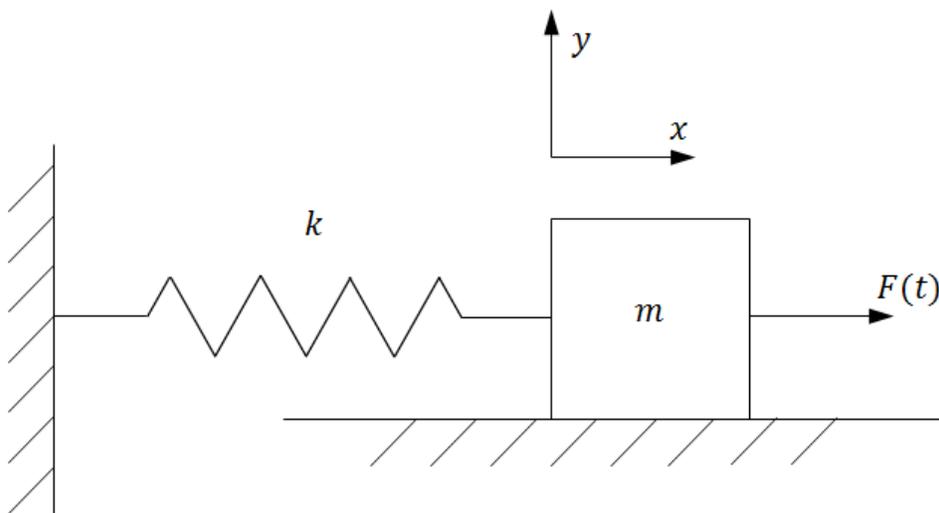
▼ Harmonic Excitation of Undamped Systems

Fig. 1: Spring- mass system with an external force.

Consider a simple mass-spring system with negligible damping on a frictionless surface. For the case of harmonic excitation, $F(t)$ has the form of a sine or cosine function of a single frequency. Here the driving force $F(t)$ will be of the form:

$$F(t) = F_0 \cdot \cos(\omega \cdot t)$$

... Eq. (1)

where F_0 is the magnitude of the force and ω is the angular frequency of the applied force. This frequency is often referred to as the input frequency, driving frequency, or forcing frequency and has units of rad/s.

The sum of the forces in the y-direction is 0, resulting in no motion in that direction. Summing forces on the mass in the x-direction yields

$$m \cdot \ddot{x}(t) + k \cdot x(t) = F_0 \cdot \cos(\omega \cdot t)$$

... Eq. (2)

Dividing through by m gives

$$\ddot{x}(t) + \omega_n^2 \cdot x(t) = f_0 \cdot \cos(\omega \cdot t)$$

... Eq. (3)

where $f_0 = \dfrac{F_0}{m}$. A solution for this type of equation can be found using the method of undetermined coefficients by summing the homogeneous solution (the solution for the case $f_0 = 0$) and a particular solution. The particular solution can often be found by assuming that it has the same form as the forcing function. That is, the oscillation of a single-degree-of-freedom system excited by $f_0 \cdot \cos(\omega \cdot t)$ is observed to be of the form

$$x_p = X \cdot \cos(\omega \cdot t)$$

... Eq. (4)

Where x_p denotes the particular solution and X is the amplitude of the forced response. Substitution of the assumed form into the equation of motion yields

$$-\omega^2 \cdot X \cdot \cos(\omega \cdot t) + \omega_n^2 \cdot X \cdot \cos(\omega \cdot t) = f_0 \cdot \cos(\omega \cdot t)$$

... Eq. (5)

Factoring out $\cos(\omega \cdot t)$ and solving for X gives

$$X = \frac{f_0}{\omega_n^2 - \omega^2}$$

... Eq. (6)

provided that $\omega_n^2 - \omega^2$ is not zero. Therefore as long as the driving and natural frequencies are not equal the particular solution will be of the form

$$x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos(\omega \cdot t)$$

... Eq. (7)

Since the system is linear, the total solution $x(t)$ is the sum of the particular solution and the homogeneous solution

$$x(t) = x_p(t) + x_h(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos(\omega \cdot t) + A \cdot \sin(\omega_n \cdot t + \phi)$$

... Eq. (8)

Recalling that $A \cdot \sin(\omega_n \cdot t + \phi)$ can be represented as $A_1 \cdot \sin(\omega_n \cdot t) + A_2 \cdot \cos(\omega_n \cdot t)$, the total solution can be expressed in the form

$$x(t) = A_1 \cdot \sin(\omega_n \cdot t) + A_2 \cdot \cos(\omega_n \cdot t) + \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos(\omega \cdot t)$$

... Eq. (9)

With constants of integration A_1 and A_2. These are determined by initial conditions. Let the initial position and velocity be given by constants x_0 and v_0. Then

$$x_0 = x(0) = A_2 + \frac{f_0}{\omega_n^2 - \omega^2}$$

<div align="right">... Eq. (10)</div>

and

$$v_0 = \dot{x}(0) = \omega_n \cdot A_1$$

<div align="right">... Eq. (11)</div>

Solving the above two equations for A_1 and A_2 and substituting these values into Eq. (9) yields the total response

$$x(t) = \frac{v_0}{\omega_n} \cdot \sin\left(\omega_n \cdot t\right) + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right) \cdot \cos\left(\omega_n \cdot t\right) + \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos\left(\omega \cdot t\right)$$

<div align="right">... Eq. (12)</div>

For example, the following plot shows the response of an undamped system with $\omega_n = 0.75$ rad/s, $x_0 = 0.03$ m and $v_0 = 0.01$ m/s to harmonic excitation of magnitude $f_0 = 0.2$ N/kg at $\omega = 2$ rad/s and $\omega = 4$ rad/s.

plot

Displacement vs. Time

Driving frequency=2 rad/s   Driving frequency=4 rad/s
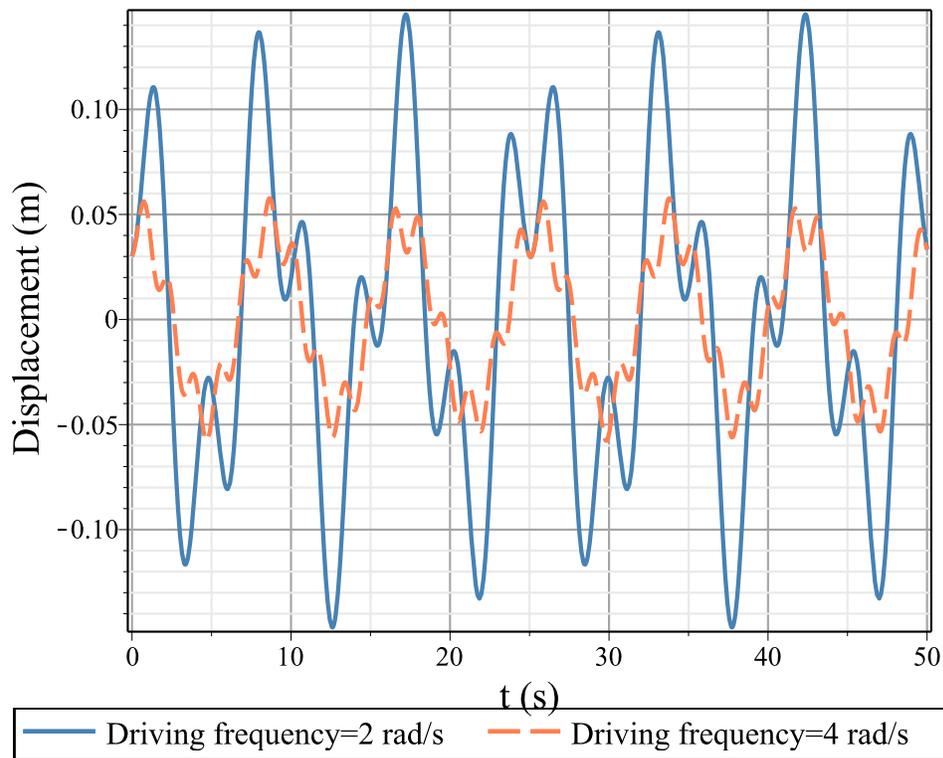
This shows that the waveform of the response can be very different for different excitation frequencies. Also, it shows that the amplitude of the motion changes with the excitation frequency. In the following plot, the gauge can be used to adjust this frequency for the initial conditions and parameters mentioned above. As the excitation frequency is brought closer to the natural frequency, the amplitude of the vibrations will become very large.

$\omega \ (rad/s)$

Displacement vs. Time

Displacement (m) vs. t (s)
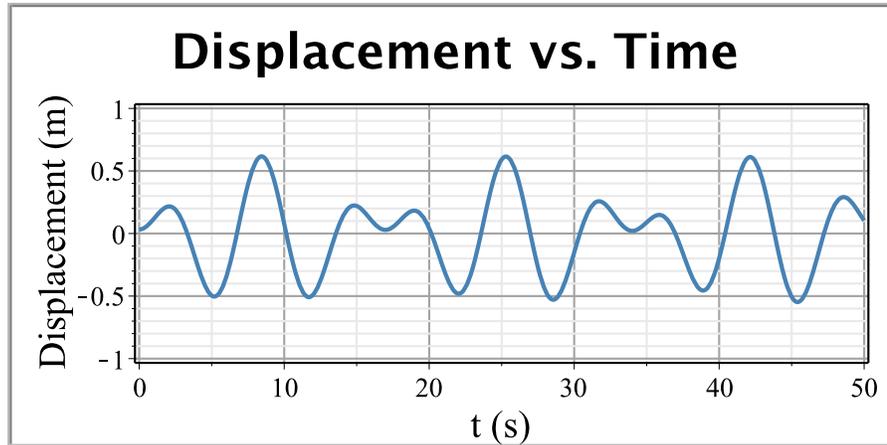
What happens when ω is near ω_n

Two important phenomena occur when the driving frequency comes close to the system's natural frequency. These two phenomena are called *beats* and *resonance*. Consider the case when $\omega_n - \omega$ becomes very small, but not equal. For zero initial conditions the response is

$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos\left(\omega_n \cdot t\right) + \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos\left(\omega \cdot t\right)$$

... Eq. (13)

which can be written as

$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cdot \left(\cos\left(\omega_n \cdot t\right) + \cos\left(\omega \cdot t\right)\right)$$

... Eq. (14)

Using simple trigonometric identities, this becomes

$$x(t) = \frac{2f_0}{\omega_n^2 - \omega^2} \cdot \sin\left(\frac{\omega_n - \omega}{2}t\right) \cdot \sin\left(\frac{\omega_n + \omega}{2}t\right)$$

<div align="right">... Eq. (15)</div>

Since $\omega_n - \omega$ is small, $\omega_n + \omega$ is large by comparison and the term $\sin\left(\frac{\omega_n - \omega}{2}t\right)$ oscillates with a much longer period than does $\sin\left(\frac{\omega_n + \omega}{2}t\right)$. Recall that the period of oscillation T is defined as

$$T = \frac{2 \cdot \pi}{\omega}$$

<div align="right">... Eq. (16)</div>

or in this case

$$T = \frac{2 \cdot \pi}{\frac{\omega_n - \omega}{2}} = \frac{4 \cdot \pi}{\omega_n - \omega}$$

<div align="right">.. Eq. (17)</div>

This gives a rapid oscillation with an amplitude that increases and decreases very slowly. This motion is called a beat. If the vibrating object was a guitar string it would produce a sound of the higher frequency which would repeatedly increase and decrease in volume over time.

For example, the response of an undamped system with $f_0 = 10$ N, $\omega_n = 10$ rad/s, and $\omega = 1.1 \cdot \omega_n$ rad/s is displayed below. The dashed line in the animation represents the term $\frac{2f_0}{\omega_n^2 - \omega^2} \cdot \sin\left(\frac{\omega_n - \omega}{2}t\right)$.

Animated plot (Click on the plot and use the toolbar that app
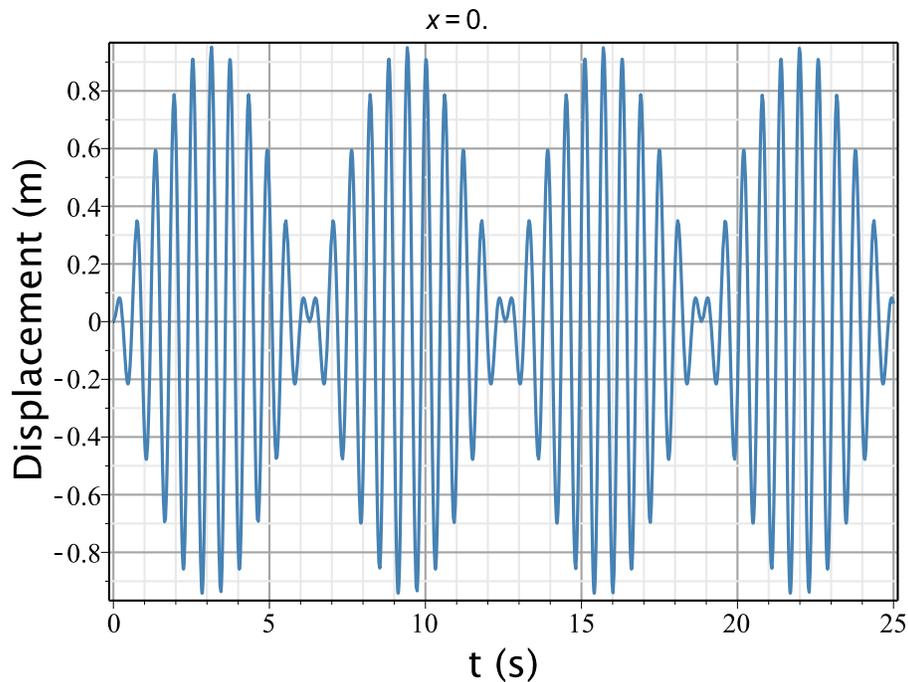
Now, let's consider the case where the driving frequency is equal to the natural frequency. Here, the solution given by Eq. (12) is no longer valid because it becomes a solution of the homogeneous equation. Therefore, for this case, we have to use a different particular solution of the form

$$x_p(t) = t \cdot X \cdot \sin(\omega \cdot t)$$

... Eq. (18)

Substituting this into Eq. (3) and solving for X yields

$$x_p(t) = \frac{f_0}{2 \cdot \omega} \cdot t \cdot \sin(\omega \cdot t)$$

.. Eq. (19)

Thus the total solution is now of the form

$$x(t) = A_1 \cdot \sin(\omega \cdot t) + A_2 \cdot \cos(\omega \cdot t) + \frac{f_0}{2 \cdot \omega} \cdot t \cdot \sin(\omega \cdot t)$$

.. Eq. (20)

Using the initial conditions x_0 and v_0 this can be written as

$$x(t) = \frac{v_0}{\omega_0} \cdot \sin(\omega \cdot t) + x_0 \cdot \cos(\omega \cdot t) + \frac{f_0}{2 \cdot \omega} \cdot t \cdot \sin(\omega \cdot t)$$
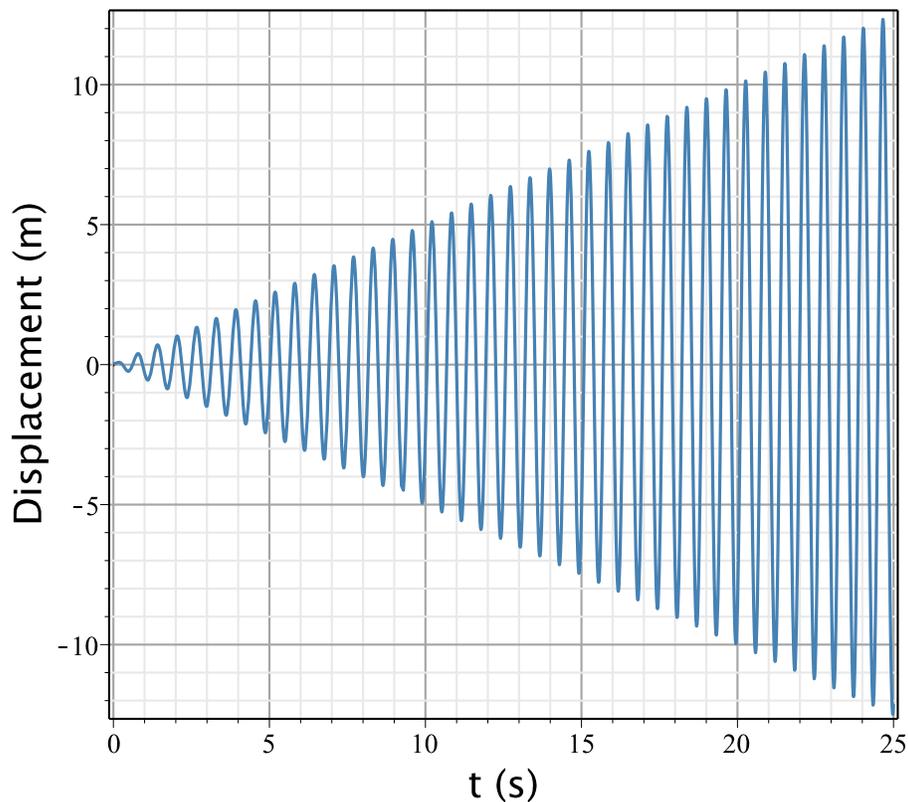
... Eq.(21)

For example, the following plot shows the response of an undamped system with $\omega_n = \omega = 10$ rad/s, $x_0 = 0.03$ m and $v_0 = 0.01$ m/s to harmonic excitation of magnitude $f_0 = 10$ N/kg. The dashed lines of the animation represent $\pm \frac{f_0}{2 \cdot \omega} \cdot t.$

Animated plot (Click on the plot and use the toolbar that app
$x = 0.$

As can be seen, the amplitude of the vibration continuously increases with time. For a mass connected to a spring this would mean that after a certain point the vibrations would no longer be in the elastic region of the spring, and the spring would deform and potentially break. For other analogous structures this would mean the same thing. It is

very important to understand this phenomena because it can be very dangerous for structures and systems.

Response to a Sine forcing function

Up till now we have assumed that the forcing function is a cosine function. If the forcing function is a sine function, then the total solution can be written as

$$x(t) = x_0 \cdot \cos\left(\omega_n \cdot t\right) + \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2} \right) \cdot \sin\left(\omega_n \cdot t\right) + \frac{f_0}{\omega_n^2 - \omega^2} \cdot \sin\left(\omega \cdot t\right)$$

... Eq. (22)

For example, the following plot shows the response of an undamped system with $\omega_n = 0.75$ rad/s, $x_0 = 0.03$ m and $v_0 = 0.01$ m/s to harmonic excitation (sine function) of magnitude $f_0 = 0.2$ N/kg at $\omega = 2$ rad/s and $\omega = 4$ rad/s.

`plot`

Displacement vs. Time

Legend:
— Driving frequency=2 rad/s
– – Driving frequency=4 rad/s
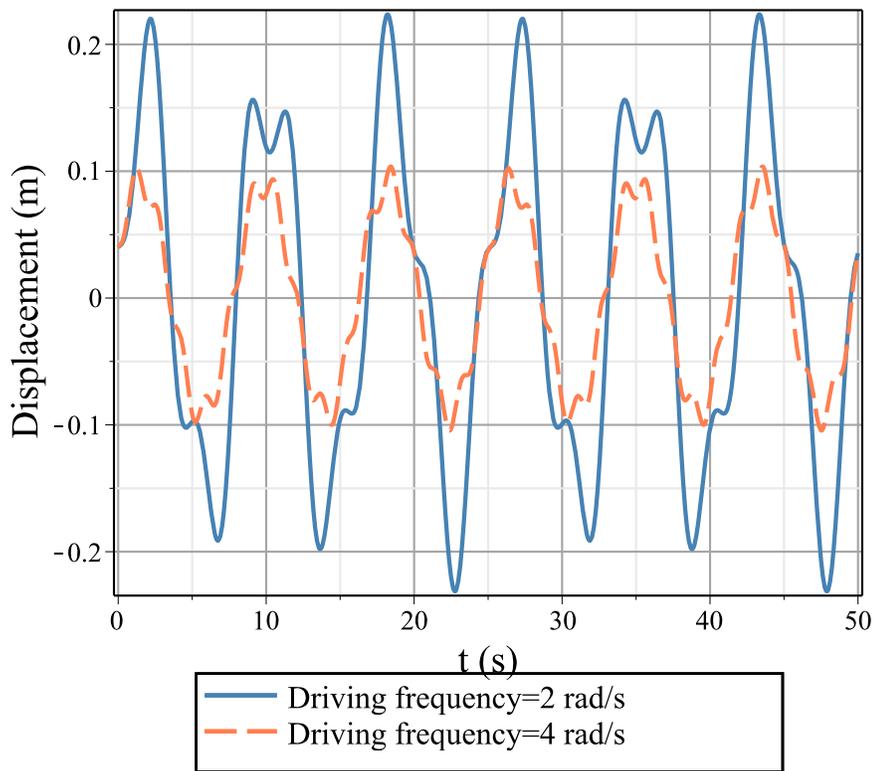
▼ Examples with MapleSim

▼ Example 1: Spring-mass oscillator

Problem Statement: Compute the total response of a spring-mass system with the values given below and plot the response for 10 seconds.

Table 1: Example 1 parameter values

Parameter	Value
k	1000

	N/m
m	10 kg
F_0	100 N
ω	8.16 rad/s
x_0	0.01 m
v_0	0.01 m/s

Analytical Solution

$restart$:

k := 10\00 :   [N/m]

m := 10 :   [kg]

F_0 := 10\0 :   [N]

ω := 8.\16 :   [rad/s]

x_0 := 0.\01 :   [m]

v_0 := 0.\01 :   [m/s]

$$\omega_n := \sqrt{\frac{k}{m}} = 10$$

$$f_0 := \frac{F_0}{m} :$$

$$x := \frac{v_0}{\omega_n} \cdot \sin\left(\omega_n \cdot t\right) + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right) \cdot \cos\left(\omega_n \cdot t\right) + \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos\left(\omega \cdot t\right) :$$

x

$$.00100\,\sin\left(10.00000\,t\right) - .28927\,\cos\left(10.00000\,t\right) + .29927\,\cos\left(8.16000\,t\right) \qquad \textbf{(2.1.1.1)}$$

$plot(x, t = 0..10)$
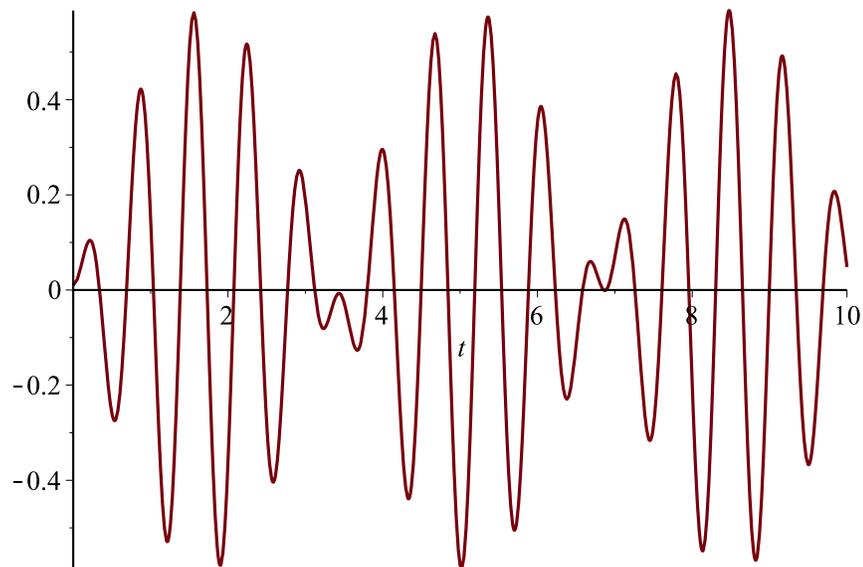
MapleSim Simulation

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
	1-D Mechanical > Translational > Common
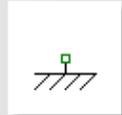
Translational Fixed	
Translational Spring	1-D Mechanical > Translational > Common
Mass	1-D Mechanical > Translational > Common
Force	1-D Mechanical > Translational > Common
Sine Source	Signal Blocks > Common
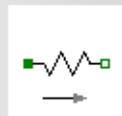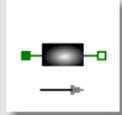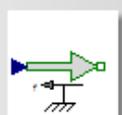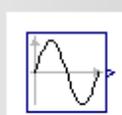
Step 2: Connect the components
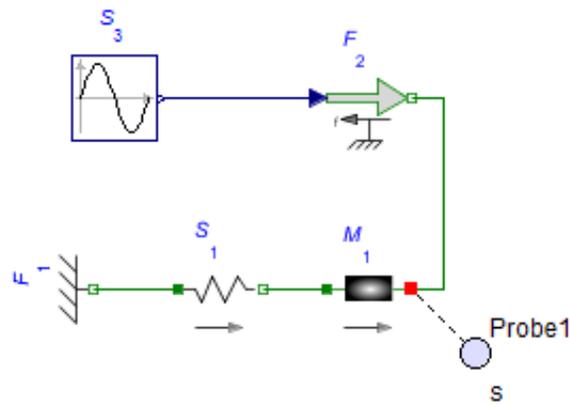
Connect the components as shown in the following diagram.

Fig. 2: Component diagram

Step 3: Set parameters and initial conditions

1. Click the **Translational Spring** component and enter **1000** *N/m* for the spring constant (c).
2. Click the **Mass** component and enter **10** *kg* for the mass (m), **0.01** *m/s* for the initial velocity (v_0) and **0.01 m** for the initial position (s_0). Select the check marks that enforce these initial condition.
3. Click the **Sine Source** component and enter **100** for the amplitude, **8.16** *rad/s* for the frequency (f) and **Pi/2** for the phase (ϕ) [since we are assuming that the excitation is a cosine function].

Step 4: Run the Simulation

1. Attach a **Probe** to the **Mass** component as shown in Fig. 2. Click this probe and select **Length** in the **Inspector** tab. This will show the position of the mass as a function of time.
2. Click **Run Simulation** (▶).

▼ Example 2: Spring-Pendulum

Problem Statement: A component of a machine is modeled as a pendulum connected to a spring (as shown on the right). This component is driven by a motor that applies a sinusoidal moment $M(t) = 10 \cos(4 \cdot Pi \cdot t)$ Nm about the axis of rotation. Derive the equation of motion and find the natural frequency of the system. The mass of the pendulum is 2kg, the length of the pendulum is 0.5m and the stiffness of the spring is 20 N/m. Assume that the rod of the pendulum has no mass. The initial angular displacement is 0.175 rad (approx. 10°) measured from the vertical and there is no initial angular velocity. Plot the response and find the maximum displacement.
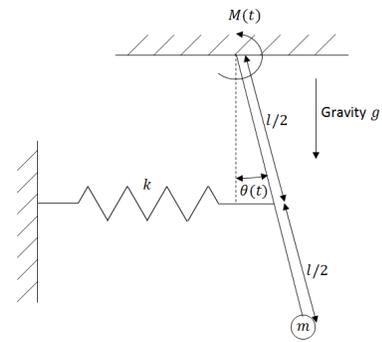

Fig. 3: Spring-pendulum example

Analytical Solution

Data:

restart :

$m := 2$:   [kg]

$l := 0.5$:   [m]

$k := 20$:   [N/m
]

$g := 9.81$:   [m/s²
]

$\theta_0 := 0.175$:   [rad]

$\omega_0 := 0$:   [rad/
s]

$\omega := 4 \cdot Pi$:   [rad/
s]

Solution:

Using the small angle approximation, we will assume that the spring stretches and compresses in the horizontal direction only. Hence the force due to the spring can be written as

$$F_k := -k \cdot \left(\frac{l}{2} \right) \cdot \theta(t) :$$

Taking the sum of the moments of force about the pivot we get

$$J_0 \cdot \ddot{\theta}(t) = F_k \cdot \left(\frac{l}{2} \right) - m \cdot g \cdot l \cdot \theta(t) + M(t) :$$

where J_0 is the moment of inertia of the pendulum. This equation can be rewritten as,

$$m \cdot l^2 \cdot \ddot{\theta}(t) + \left(k \cdot \left(\frac{l}{2} \right)^2 + m \cdot g \cdot l \right) \cdot \theta(t) = 10 \, \cos(4 \cdot Pi \cdot t) :$$

or

$$\ddot{\theta}(t) + \frac{\left(\frac{k}{4} + \frac{m \cdot g}{l} \right)}{m} \cdot \theta(t) = \frac{10}{m \cdot l^2} \, \cos(4 \cdot Pi \cdot t) :$$

Comparing this equation to the form of Eq. (3) we can conclude that the natural frequency of the system will be

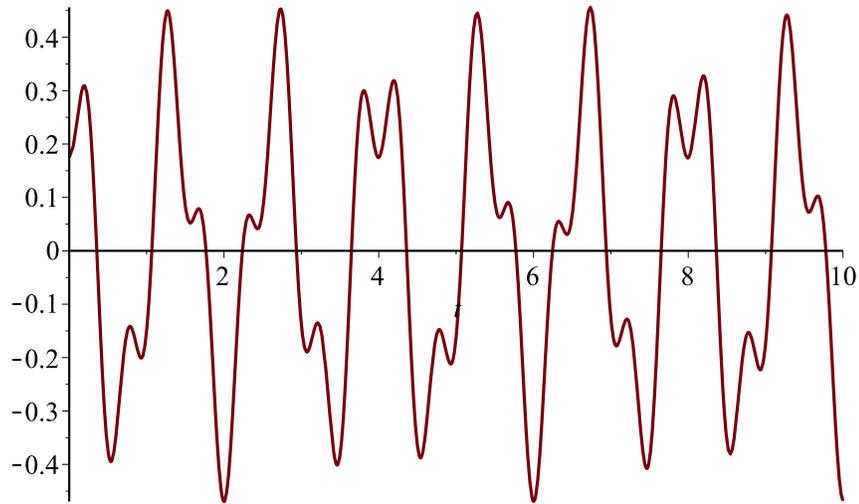$$\omega_n := \sqrt{\frac{\frac{k}{4} + \frac{m \cdot g}{l}}{m}} = 4.70$$

and that the solution will be

$$\theta := \left\{ \frac{\omega_0}{\omega_n} \cdot \sin\left(\omega_n \cdot t \right) + \left(\theta_0 - \frac{\frac{10}{m \cdot l^2}}{\omega_n^2 - \omega^2} \right) \cdot \cos\left(\omega_n \cdot t \right) + \frac{\frac{10}{m \cdot l^2}}{\omega_n^2 - \omega^2} \cdot \cos\left(\omega \cdot t \right) \right\}$$

$$\left\{ \left(0.175 - \frac{20.00000000}{-16 \, \pi^2 + 22.12000000} \right) \cos(4.703190407 \, t) \right. \qquad \textbf{(2.2.1.1)}$$

$$\left. + \frac{20.00000000 \, \cos(4 \, \pi \, t)}{-16 \, \pi^2 + 22.12000000} \right\}$$

$plot(\theta, t = 0 .. 10)$

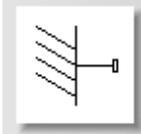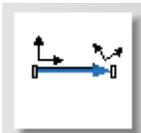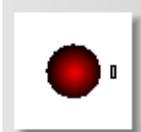From this plot, we can see that that maximum displacement is approximately 0.45 rad.
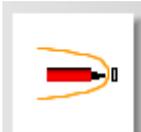
MapleSim Soultion

Constructing the model

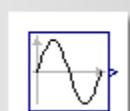**Step1**: Insert Components

Drag the following components into the workspace:

Table 3: Example 2 components and locations

Component	Location
Fixed Frame (2 required)	Multibody > Bodies and Frames
	Multibody > Joints and Motions

Revolute	
Rigid Body Frame (2 required)	Multibody > Bodies and Frames
Rigid Body	Multibody > Bodies and Frames
Spherical Geometry	Multibody > Visualization
Path Trace	Multibody > Visualization
Cylindrical Geometry	Multibody > Visualization
	Multibody > Forces and Moments
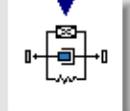
Translational Spring Damper Actuator	
Spring Geometry	Multibody > Visualization
Applied World Moment	Multibody > Forces and Moments
Sine Source	Signal Blocks > Common > Sine Source

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for).
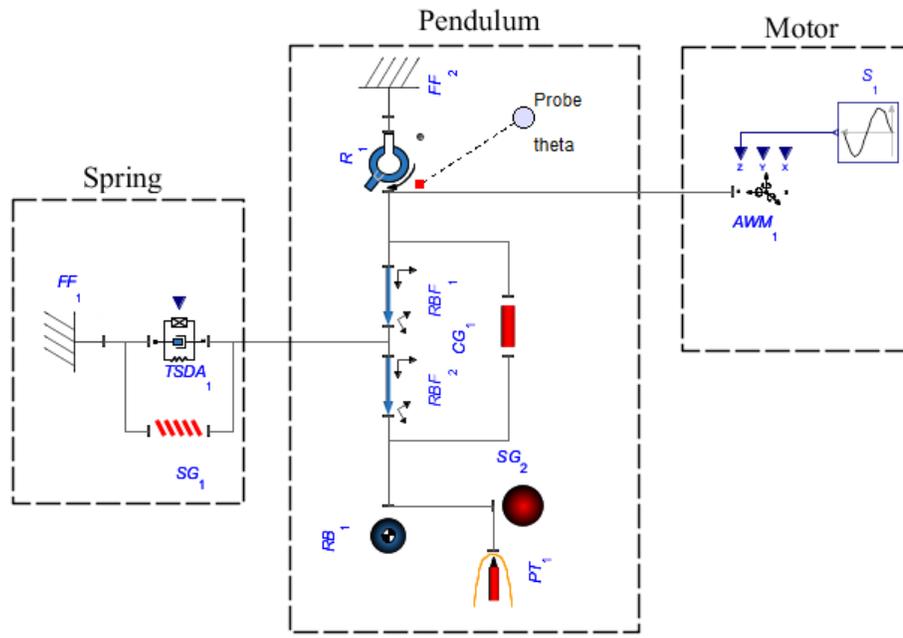
Fig. 4: Example 2 Component diagram

Step 3: Set up the Pendulum

1. Click the **Revolute** component and enter **0.175** *rad* for the initial angle (θ_0) and select **Strictly Enforce** in the drop down menu for the initial conditions ($IC_{\theta,\,\omega}$). The axis of rotation ($\hat{e}_1$) should be left as the default axis **[0,0,1]**.

2. Enter **[0,-0.25,0]** for the x,y,z offset ($\bar{r}_{XYZ}$) of both the **Rigid Body Frames**.

3. Enter **2** *kg* for the mass (*m*) of the **Rigid Body Frame**.

Step 4: Set up the Spring

1. Click the **Fixed Frame** component connected to the **TSDA** (**FF$_1$** in the diagram) and enter **[-0.25,-0.25,0]** for the x,y,z offset ($\bar{r}_{XYZ}$).

2. Click the **TSDA** component and enter **20** *N/m* for the spring constant (K_{spring}). Also, enter **0.25** *m* for the unstretched length (l_0) to correspond to the location of the **Fixed Frame**.

Step 5: Set up the external sinusoidal moment (Motor)

1. Click the **Sine Source** component and enter **10** for the amplitude, **4*Pi** *rad/s* for the frequency (*f*) and **Pi/2** for the phase (φ) [the external force is a cosine function].
2. Connect the output of the **Sine Source** component to the **z** input of the **Applied World Moment** component.

Step 6: Set up the visualization (Inserting the **Visualization** components is optional)

1. Click the **Cylindrical Geometry** component and enter a value around **0.01** *m* for the radius.
2. Click the **Spherical Geometry** component and enter a value around **0.05** *m* for the radius.
3. Click the **Spring Geometry** component, enter a number around **10** for the number of windings, enter a value around **0.02** *m* for **radius1** and enter a value around **0.005** *m* for **radius2**.

Step 7: Run the Simulation

1. Click the **Probe** attached to the **Revolute** joint and select **Angle** to obtain a plot of the angular position vs. time.
2. Click **Run Simulation** (▶).

Since the analytical solution makes a small angle approximation, there will be a very slight variation in the results of this simulation and the results of the analytical method. From the plot generated using the analytical approach, the maximum amplitude is found to be approximately 0.45 rad. And, from the plot generated using the simulation, the maximum amplitude is found to be approximately 0.47 rad.

The following image shows the plot obtained from the simulation that shows the angle with respect to time.
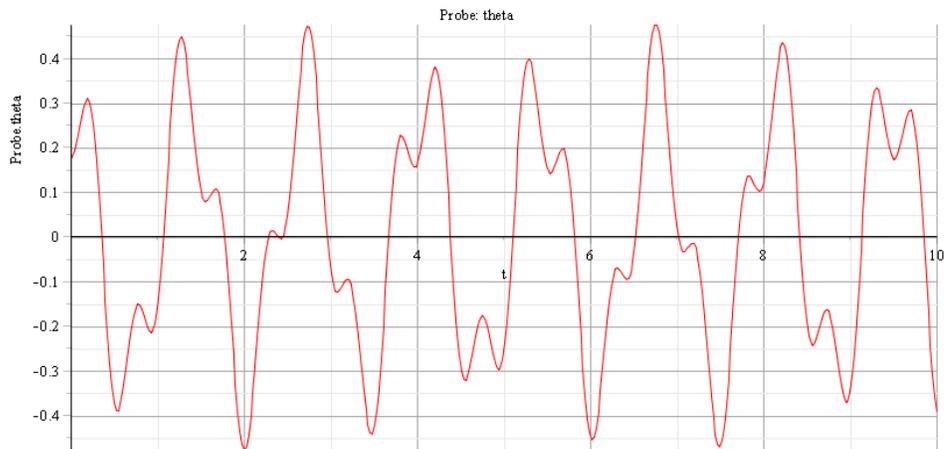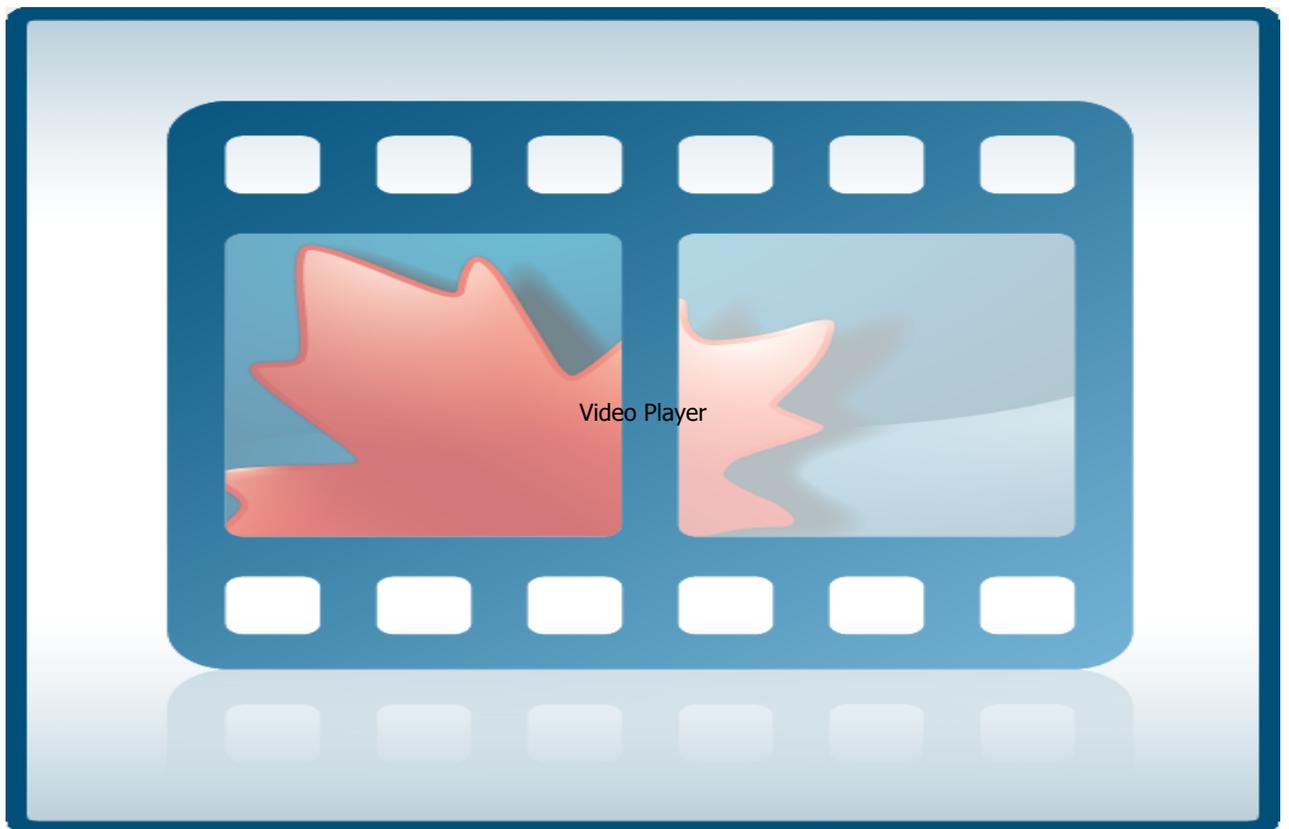
Fig. 5: Simulation results (theta vs. t)

The following video shows the visualization of the simulation

Video 1: Visualization of the spring-pendulum simulation

Reference:

D. J. Inman. "Engineering Vibration", 3rd Edition. Upper Saddle River, NJ, 2008, Pearson Education, Inc.